Characteristic distributions of finite-time Lyapunov exponents

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We study the probability densities of finite-time or *local* Lyapunov exponents in low-dimensional chaotic systems. While the multifractal formalism describes how these densities behave in the asymptotic or long-time limit, there are significant finite-size corrections, which are coordinate dependent. Depending on the nature of the dynamical state, the distribution of local Lyapunov exponents has a characteristic shape. For intermittent dynamics, and at crises, dynamical correlations lead to distributions with stretched exponential tails, while for fully developed chaos the probability density has a cusp. Exact results are presented for the logistic map, $x \rightarrow 4x(1-x)$. At intermittency the density is markedly asymmetric, while for ''typical'' chaos, it is known that the central limit theorem obtains and a Gaussian density results. Local analysis provides information on the variation of predictability on dynamical attractors. These densities, which are used to characterize the *nonuniform* spatial organization on chaotic attractors, are robust to noise and can, therefore, be measured from experimental data. [S1063-651X(99)10208-3]

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I. INTRODUCTION

The statistics of distributions of Lyapunov exponents (sometimes called stretch exponents) have been studied in a number of physical situations ranging from turbulent flows [1] to Hamiltonian dynamics (in many-particle systems [2] and conservative mappings [3]), and are related to generalized dimensions and entropies [4,5].

Lyapunov exponents provide a quantitative characterization of dynamics: for a dynamical system in D dimensions, the D Lyapunov exponents of an orbit measure the rate at which volume elements in the phase space expand or contract along the orbit, in the different directions. A positive Lyapunov exponent (LE) signifies exponential divergence of trajectories in the given direction, and is associated with chaotic dynamics, while negative LEs are associated with stable motion, when nearby trajectories converge. Although the exponents are global or asymptotic quantities, it is often instructive to examine the distribution of values that the LE may take locally, namely, over finite-time segments along a given trajectory [5–7]. If the underlying attractor is nonuniform, on a chaotic trajectory the *local* LE can be negative within a finite-time interval. Similarly, on a nonchaotic trajectory the local LE can take positive values over finite-time intervals [8]. These considerations are of additional relevance for Lyapunov exponents computed from experimental data [9], since these are, naturally, only finite-time exponents.

In this paper, we study the statistics of finite-time or local LEs in low-dimensional dissipative dynamical systems. Local Lyapunov exponents, which are defined in Eq. (6) below, depend on initial conditions, unlike the asymptotic or global Lyapunov exponent. The distribution of values that the local exponents take depends, in a characteristic manner, on the nature of the dynamical state; our present focus is on characterizing the different distributions that are obtained for different dynamical attractors.

We find that the characteristic densities of Lyapunov exponents fall in distinct classes depending on the nature of the attractor. Although the local Lyapunov exponent (LLE) distributions are stationary, they depend on the time interval over which the finite-time LE is computed. For very short times, the distributions keep changing shape and are difficult to classify, while in the asymptotic limit, as the time interval $\rightarrow \infty$, all distributions must eventually collapse to a δ function centered on the global LE. The manner in which this happens is usually described by the multifractal formalism [5], but here we address the important corrections to scaling that can be obtained for finite times.

Some aspects of such distributions have been studied previously. For ''typical'' chaotic dynamics, when correlations die out exponentially rapidly, the central-limit theorem holds for a number of averaged quantities, including local Lyapunov exponents [5,7]. Thus, the density is a Gaussian function, whose width depends on the length of the time interval over which the exponents are computed. For intermittent systems on the other hand, it is well known that correlations die out very slowly; this can lead to a power-law scaling for several quantities such as the Lyapunov exponent or the diffusion constant [10]. Benzi *et al.* also studied intermittency in more detail and observed a deviation from the normal distribution in quantities such as the fluctuations of the response function [11].

We extend the analysis of finite-time Lyapunov exponents to the particular case of fully developed chaos and intermittency, where dynamical correlations persist over long times. This leads to significant departures from the simple central limit behavior, and the resulting distributions are quite distinct from the normal density, typically having exponential or stretched exponential tails.

An example, which can be solved exactly, is the commonly studied logistic map at the Ulam point, namely, $x \rightarrow 4x(1-x)$, for which we obtain an analytic form of the probability density for all times. The same distribution occurs for all parameter values where there is a boundary crisis, and thus appears to be quite general.

We also treat the case of intermittent dynamics in some detail. In all instances of intermittency, the dynamics

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switches between two or more distinct types of behavior. The distribution of LLEs that arise in such a situation can be shown to have components arising separately from these individual behaviors.

Our main results are presented in Sec. II, where we discuss the different types of distributions for finite-time LEs. Our studies have been mainly on simple dynamical systems such as the logistic mapping, but the results we obtain appear to be more generally valid: these distributions can be seen in a variety of systems (both mappings and flows). This is followed by a summary and conclusions in Sec. III.

II. CHARACTERISTIC DENSITIES OF FINITE-TIME EXPONENTS

For generality, consider a D-dimensional discrete nonlinear system [12]

$$\mathbf{X}_{n+1}=F_{\{\alpha\}}(X_n), \tag{1}$$

where $\mathbf{X}\in\mathbf{R^D}$ and $\{\alpha\}$ are a set of parameters. There are D Lyapunov exponents that are defined by considering a set of orthonormal D-dimensional vectors $\hat{\mathbf{e}}^m$, $m=1,\ldots,D$, and examining their evolution under the effect of the tangent mapping, which is determined by the Jacobian of F, namely, $JF(\mathbf{X})$. Defining

$$\mathbf{e}_j^m=[JF_{\{\alpha\}}(\mathbf{X}_{j-1}),\hat{\mathbf{e}}_{j-1}^m], \tag{2}$$

the Lyapunov exponents are

$$\Lambda^m=\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\ln\|\mathbf{e}_j^m\|, \quad m=1,2,\ldots,D. \tag{3}$$

The vectors $\hat{\mathbf{e}}^m$ are reorthonormalized along the trajectory, and the subscript j here refers to the time. Stretch exponents are the logarithms of the ratios by which the vectors expand (or contract) along the D directions,

$$\lambda_1^m(j)=\ln\|\mathbf{e}_j^m\|, \tag{4}$$

and this helps to define the mth finite-time Lyapunov exponent in a time interval of length N as

$$\lambda_N^m=\frac{1}{N}\sum_{j=1}^{N}\ln\|\mathbf{e}_j^m\|. \tag{5}$$

In the remainder of this paper only $m\equiv 1$, the largest Lyapunov exponent, is considered, so the superscript index will be omitted henceforth in order to simplify notation. Local Lyapunov exponents are calculated along a trajectory that is divided into segments of length N. By $\lambda_N(i)$ is meant the N-step Lyapunov exponent calculated from the ith segment, and clearly

$$\lambda_N(k)=\frac{1}{N}\sum_{j=N(k-1)+1}^{kN}\lambda_1(j). \tag{6}$$

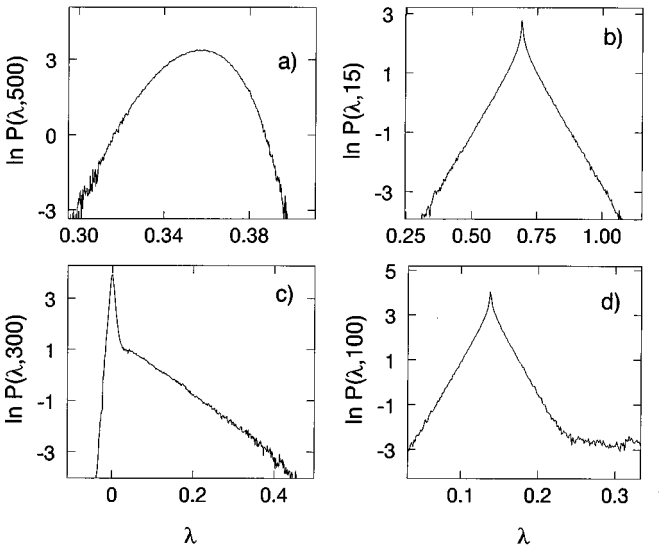

FIG. 1. Characteristic probability densities that arise in chaotic dynamical systems, corresponding to (a) ''typical'' chaos, (b) fully developed chaos, (c) intermittency, and (d) crisis-induced intermittency. Numerical results are for $P(\lambda,N)$ for the logistic map with (a) $\alpha=3.7$, (b) $\alpha=4$, (c) $\alpha=1+\sqrt{8}-10^{-6}$, and (d) $\alpha=3.7\ 447\ 104$. The local Lyapunov exponents are calculated from N step segments with 10^6 iterations. Different values of N are chosen for the three cases, in (a) the Gaussian nature becomes apparent at N around 10 or so, the cusp in (b) is already evident for N around 8 and survives for all N, while the asymmetry in the distribution (c) and (d) appears to persist for all N. Note that for (d), N should be a multiple of the periodicity of the window.

The asymptotic Lyapunov exponent $\Lambda\equiv\lambda_\infty$ does not depend on initial conditions with probability 1 but λ_N does. The *probability density* of local Lyapunov exponents, which is a stationary quantity, is defined as

$$P_\alpha(\lambda,N)d\lambda\equiv\text{Probability that } \lambda_N \text{ takes a value between } \lambda \text{ and } \lambda+d\lambda. \tag{7}$$

If the stretch exponents of a system are considered to be random variables since the dynamics is chaotic, then the finite-time LEs should obey the central-limit theorem, and the distribution function can be written in the general form [7]

$$P(\lambda,N)\sim\frac{1}{[2\pi N\Phi'']^{1/2}}\exp -N\Phi(\lambda), \tag{8}$$

where $\Phi(\lambda)$ is a *convex* function with minimum at $\lambda=\Lambda$. Expanding Φ to second order gives the Gaussian density

$$P_\alpha(\lambda,N)\sim\exp-(\lambda-\Lambda)^2/2\sigma, \tag{9}$$

with variance $\sigma^2\propto 1/N$. Indeed this argument has been used very effectively to analyze finite-time LEs on so-called ''typical'' chaotic attractors [5,7,10] as shown in Fig. 1(a), where the Gaussian nature of the density is evident. However, for other dynamical attractors, $P(\lambda,N)$ can be quite different, as shown in Figs. 1(b)–1(d). Indeed, the departure

of $\Phi(\lambda)$ from a polynomial function with quadratic maximum has been used to characterize the state and study the persistence of correlations [3,13,14].

Regardless of the behavior of $P(\lambda,N)$ for small N, the local LEs eventually converge to the global exponent, $\lim_{N\to\infty}P_{\alpha}(\lambda,N)\to\delta(\Lambda-\lambda)$. However, for sufficient large N (but still far from the limit $N\to\infty$) the characteristic distributions depend on the details of the dynamics, so that the approach to the limit is distinctive. In the following subsections we discuss the particular cases of fully developed chaos and intermittency in detail.

A. Fully developed chaos

The case of fully developed chaos in a system such as the logistic mapping,

$$x_{n+1}=\alpha x_n(1-x_n) \tag{10}$$

at $\alpha=4$ can be analyzed in detail since the invariant density is known exactly. The Lyapunov exponent for this system is $\Lambda=\ln 2$, and the invariant density, which can be obtained by solving the appropriate Frobenius-Perron equation, is [15,16]

$$\rho(x)=1/\pi\sqrt{x(1-x)},\quad x\in[0,1]. \tag{11}$$

The one-step Lyapunov exponent,

$$\lambda_1(i)\equiv\ln|\alpha(1-2x_i)| \tag{12}$$

itself obeys the mapping

$$\lambda_1(i+1)=\ln|[\exp\{2\lambda_1(i)\}-\alpha^2+2\alpha]/2|. \tag{13}$$

$P(\lambda,1)$ is merely the invariant density for this mapping, and by using Eq. (11), one obtains

$$P(\lambda,1)=\frac{2\exp(\lambda-\ln 4)}{\pi\sqrt{1-\exp[2(\lambda-\ln 4)]}},\quad -\infty\leqslant\lambda\leqslant\ln 4. \tag{14}$$

$P(\lambda,N)$ can be calculated recursively since λ_N is known in terms of λ_1 through Eq. (6). Since λ_1 is known in terms of x_i [Eq. (12)], it is possible to re-express Eq. (6) as

$$\exp(N\lambda_N)-G(x)=0, \tag{15}$$

where G is a polynomial function of x of order 2^N-1. From this it directly follows that

$$P(\lambda,N)=N\exp(N\lambda)\sum_{\text{roots}}\frac{\rho(x)}{|G'(x)|}, \tag{16}$$

the sum being over all real roots, $x(\lambda)$, of Eq. (15) at given $\lambda\in[-\infty,\ln 4]$. Since the polynomial $G(x)$ is of odd order, there is always one real root. For sufficiently small λ all roots are real; with increasing λ they leave the real axis in pairs, each such point giving rise to a singularity since the derivative $dG(x)/dx$ vanishes there.

In principle, Eq. (16) provides an exact solution for the invariant density of finite-time Lyapunov exponents for the logistic mapping. Furthermore, the same technique can be applied to obtain invariant densities for other systems if $\rho(x)$ is known, though the form given above, namely, Eq. (16), is not particularly transparent.

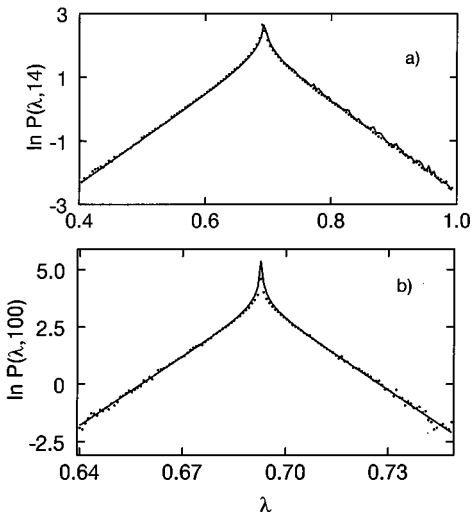

FIG. 2. (a) Comparison at $\alpha=4$ of the numerical results (dots) for the density $P(\lambda,14)$, with the analytic expression, Eq. (16), (solid line). (b) Numerical results (dots) compared with the approximate density, Eq. (17) (solid line), for $N=100$.

For this mapping, though, based on the analysis for small N, we conjecture the following asymptotic expression for the probability density of the N-step exponent,

$$P(\lambda,N)=\frac{N}{\pi}\frac{\exp(-N|\lambda-\Lambda|)}{[1-\exp(-2N|\lambda-\Lambda|)]^{1/2}},\quad -\infty\leqslant\lambda\leqslant\ln 4. \tag{17}$$

The main feature to be noted here is that there is a cusp at $\lambda=\Lambda$, which does not vanish even as $N\to\infty$ although the range of λ where it is significant decreases sharply with N. Outside this range, the function behaves essentially like an exponential. Equation (17) thus provides finite-size corrections to the expressions derived earlier by Grassberger, Badii, and Politi (see Eq. 4.9 in Ref. [5]).

Both the results, namely, Eq. (16) or Eq. (17) can be verified numerically and can be shown to hold to a high level of accuracy. Shown in Fig. 2(a) are the (numerical) experimental distributions for the case of $N=14$, which very closely matches both the exact (implicit) distribution, Eq. (16), as well as the asymptotic result, Eq. (17) [Fig. 2(b)]. Evaluation of the former expression requires the determination of the roots of the corresponding polynomial (we use the Newton-Raphson procedure). Shown in Fig. 3 is the solution for the case of $N=3$, when the polynomial $G(x)$ is of order 7. The divergences in the distribution $P(\lambda,3)$ occur as pairs of real roots merge. For the specific case of the logistic mapping, as N increases the largest number of singularities accumulate near $\Lambda=\ln 2$. The asymptotic form, Eq. (17), also gets progressively more accurate with increasing N [see Fig. 2(b)].

This form of the density is not restricted to the logistic mapping at $\alpha=4$, but is also seen in a number of mappings, which have fully developed chaos. Even in the logistic map-

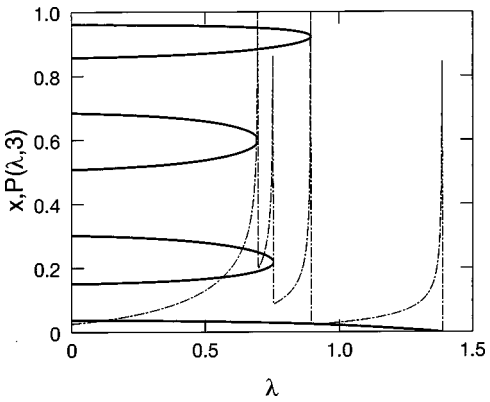

FIG. 3. The real roots (solid line) and $P(\lambda,3)$ (dash-dotted line) for the logistic map with $\alpha=4$. The singularities in the density occur when a pair of real roots become complex. $P(\lambda,3)$ has been rescaled for clarity.

ping, it occurs at *all* parameter values corresponding to widening crises [17], when the attractor is a rescaled image of the attractor at $\alpha=4$.

We have also examined the dependence of the variance of the distribution on N. Typically, $\sigma^2 \propto 1/N^{\gamma}$ for these distributions since they narrow with increasing N, going, in the limit of $N\to\infty$, to a δ function. For the case of a Gaussian density, $\gamma=1$, while for the exponential density the variance decreases more rapidly, and $\gamma=2$. Our results for the variance are shown in Fig 4.

B. Intermittency

Intermittent dynamics is characterized by a long-range temporal persistence of correlations [18], evidenced for example by the existence of power-law dependence of a number of quantities on the parameters. The question of the distribution of the local Lyapunov exponent for intermittent chaos has been explored previously by Benzi *et al.* [11], who showed that there are significant departures from the Gaussian distribution.

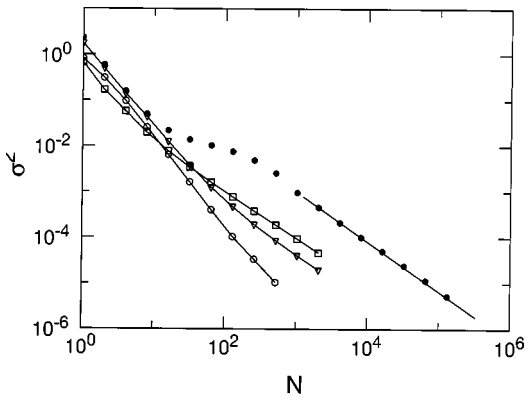

FIG. 4. (a) The variance as a function of N for ''typical'' chaos ($\square$), fully developed chaos ($\bigcirc$), and crisis-induced intermittency (∇). The exponents characterizing the decay are 1.12, 1.95, and 1.51, respectively. For intermittent chaos (filled circles) there is a crossover: the exponents in different ranges of N going from exponential limit 1.85 to the Gaussian limit 1.09 at large N.

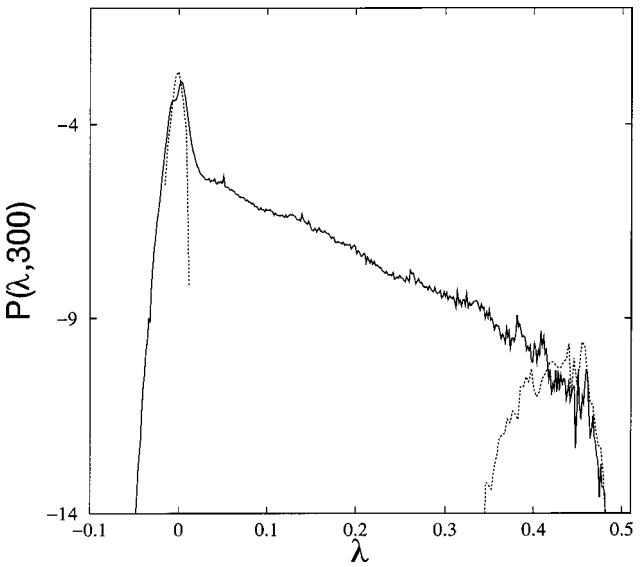

FIG. 5. Near the intermittent transition at $[\alpha=1+\sqrt{(8)}-10^{-6}]$, where there are long crossovers, the two components of the density $P(\lambda,300)$ (dotted lines) are compared with the total density (solid line). For this state, $\lambda_p=-0.000\ 15$ and $\lambda_*=0.03$. Note that the amplitudes of the individual densities have been appropriately scaled to depict clearly the manner in which they contribute to the total density.

For the case of intermittency, the characteristic density of the local Lyapunov exponent appears to be a combination of a normal density and a stretched exponential tail, an example of which is shown in Fig. 1(c). For $\lambda\leqslant\lambda_*$, the density is a Gaussian, while above λ_*, the dependence is

$$P(\lambda,N)\approx\exp[-N^{\delta}(\lambda-\lambda_*)], \quad \lambda>\lambda_*. \tag{18}$$

Since the exponent $\delta<1$, the exponential tail decays extremely slowly with N. At $\lambda=\lambda_*$, there is a crossover between the Gaussian and the stretched exponential density, which results in a completely asymmetric density about the mean. Similar distributions arise for all intermittent dynamical states, including the case of nonchaotic dynamics [19].

One way of understanding the above (phenomenological) expression for the density is to note that in all intermittent dynamics, the motion switches between two types of states. For each of these different dynamical states, the local Lyapunov exponents have a Gaussian distribution centered at different values of λ, and with different amplitudes, and the stretched exponential behavior interpolates between them.

The example for which data is presented here in Fig. 1(c) and in Fig. 5 is the Type-I intermittency near the tangent bifurcation in the logistic mapping. This dynamics can be naturally separated into laminar regions (that stay close to the incipient period-3 orbit) and chaotic bursts. Finite-time Lyapunov exponents can be separately computed for trajectories that stay entirely in the laminar phase and entirely in the chaotic phase: these give the normal densities shown by the dotted lines in Fig. 5. Trajectory segments that visit both the components of the intermittent motion contribute to the stretched exponential tail; with increasing N, the purely chaotic component is more difficult to identify, since there are fewer segments that are of duration longer than N. The

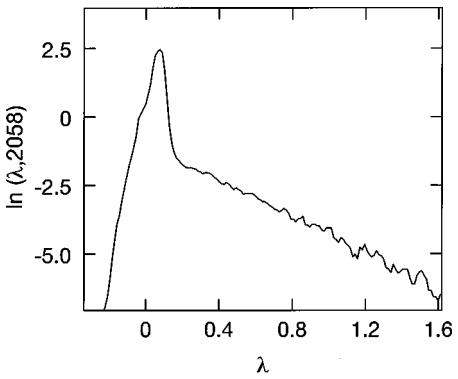

FIG. 6. For the Lorenz equations $\dot{x}=a(y-x)$, $\dot{y}=x(r-z)-y$, and $\dot{z}=xy-bz$, where a, b, and r are parameters, the density $P(\lambda,2048)$ near the intermittency transition at $a=10$, $b=8/3$, and $r=166.880\ 154\ 8$. Here λ is the largest nonzero Lyapunov exponent, and the integration step size is 0.02 natural units.

stretched exponential tail thus decreases with increasing N, and the distribution eventually collapses to a delta function.

This behavior is generic at all intermittencies. For the mapping

$$x_{n+1}=x_n+cx_n^2, \tag{19}$$

which has been extensively studied within the thermodynamic formalism in the context of Type-I intermittency [20], we observe a similar decomposition of the overall density of LLEs to a superposition of two independent Gaussians with stretched exponential interpolation between the two. Furthermore, we have also examined a number of higher-dimensional maps and flows, and find that at all intermittent dynamics, including the cases of forced systems or of nonchaotic dynamics [19,21], non-Gaussian stretched exponential tails are seen. An example shown in Fig. 6 is the density $P(\lambda,2048)$ for the largest nonzero Lyapunov exponent in the Lorenz system (see the figure caption for a definition of the dynamical system) with the parameters chosen to correspond to intermittent dynamics.

The case of crisis-induced intermittency [7] (just beyond widening crises, for instance) is similar, with the two independent densities being, respectively, the exponential cusp, namely, Eq. (17) for the component for the precrisis chaotic attractor, and a Gaussian density, which corresponds to the widened chaotic attractor. The density shown in Fig. 1(d) for the period-five widening crisis in the logistic mapping can be analyzed in a manner very similar to the case illustrated in Fig. 5.

The variance for these distributions decrease somewhat faster than for the Gaussian, namely, $\sigma^2\propto 1/N^{\gamma}$ with $\gamma\equiv 2\delta>1$. With increasing N, the exponent however changes, eventually reaching the Gaussian limit, $\gamma\approx 1$ (see Fig. 5).

III. SUMMARY AND DISCUSSION

A major motivation for the present paper has been the realization that local Lyapunov exponent distributions have characteristic forms depending on the nature of the attractors, and that these provide an additional and important dynamical characterization of the chaotic state of a system.

We have mainly focused on the cases of fully developed chaos, crises, and intermittencies—namely those attractors for which the density shows a marked departure from the simple Gaussian form. This is indicative of significant finite-size corrections to the multifractal formalism [22]. All these cases show exponential tails albeit for different reasons [23]. For fully developed chaos in the logistic mapping we obtain the (in principle) exact expression for the probability density as well as an asymptotic approximate form. These densities are seen at all parameter values corresponding to interior crises, and thus are quite common.

The case of intermittency was analyzed in detail, and the density seen there was shown to arise from individual densities corresponding to the different components of the motion (laminar and chaotic, say) with the stretched exponential tail corresponding to interpolation between these two.

We have verified the generality of our results for a number of other dynamical systems, for example, higher-dimensional mappings such as the Hénon system or flows such as the Duffing oscillator, the forced damped pendulum, the Lorenz equations, etc. These densities are quickly attained, and are maintained even as N increases, and for fairly high levels of additive noise [24]. This is of particular relevance when analyzing experimental data.

The logistic map was used here for illustration since exact results can be obtained for at least one parameter value, but the probability density for finite-time Lyapunov exponents in any system can be obtained via Eq. (16) so long as the invariant measure is known [25]. In this regard, it is interesting to note that recently, Pingel *et al.* [26] have described a general inversion technique whereby a class of one-dimensional maps having a prescribed invariant density can be constructed.

The characteristic forms for the density that we have described are found in a variety of systems, including those where the dynamics is not chaotic. When a system is forced quasiperiodically, chaotic attractors can be transformed to strange nonchaotic attractors [21]. These attractors are fractal, but the largest Lyapunov exponent is zero or negative. Because of the spatial fractality, though, for short times the local Lyapunov exponents can be positive, and the distributions again fall into the classes that we have seen here for chaotic systems [27]. The phenomenon of high-stretch tails also appears to be very general. In recent work Calvo and Labastie [28] have examined the distribution of local Lyapunov exponents in a conservative system, namely, in a 19-particle cluster simulation study. They observe that when the dynamics is intermittent [29], the Lyapunov exponent density has a stretched exponential tail.

Local analysis can be more revealing of the nature of the dynamics than global quantities, and this is an issue when predictability is of concern. For instance, in applications that aim to predict future behavior based on time-series data (for example in atmospheric sciences or in economics), atypical or extreme behavior, which contributes to the stretched tails, is a serious bottleneck. Since the largest Lyapunov exponent can be extracted from time-series data through standard techniques, examination of the finite-time distributions can give more insight into the dynamics than the extraction of a single exponent.

We conclude with a few general comments. Except for

fully developed chaos when the distributions can be derived for all times, the present study does not examine the case of very short times when the distributions are atypical and it is not clear if they are stationary. The characteristic behavior becomes apparent for times that are not too short, and persists thereafter. Thus, local Lyapunov exponent densities provide *quantitative* distinctions among different chaotic attractors. Exponential tails are characteristic of fully developed chaos and intermittency: whether these are related to analogous distributions that arise in turbulent flows [30] or stretched-exponential tails in relaxation phenomena is an interesting open question.

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